The evaluation of the probability of rock wedge failure using the point estimate method and maximum likelihood estimation method

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Abstract: In a probabilistic analysis of rock slope stability, the Monte Carlo simulation technique has been used to evaluate the probability of slope failure. While the Monte Carlo simulation technique has many advantages, the technique requires complete information for the random variable in stability analysis, but it is difficult to obtain the complete information from the field investigation in practice. The information on the random variable is usually limited due to the limitation of sampling numbers. This is why the approximation method for reliability analysis has been proposed. The approximation method such as first order second moment method (FOSM) and point estimate method (PEM) requires only the mean and standard deviation of random variable and, therefore, it is easy to utilize when the information is limited. Usually, a single closed form of formula for the evaluation of factor of safety is needed for the approximation method. However, the stability analysis method of wedge failure is complicated and cumbersome, and does not provide the simple equation for the evaluation of factor of safety. Consequently, the approximation method is not appropriate for using in the wedge failure. In order to overcome the limitation, the simple equation, obtained from maximum likelihood estimation method for wedge failure, is utilized to calculate the probability of failure. The simple equation for direct estimation of safety factors for wedge failure has been empirically derived from failed and stable cases of slope using the maximum likelihood estimation method. The developed technique has been applied to a practical example and the results from the developed technique are compared to the results from the Monte Carlo simulation technique.

Résumé: Pour l'analyse probabilistique de la stabilité du talus rocheux, la technique de simulation de Monté Carlo se servait à évaluer la probabilité de la rupture du talus. Tandis que la technique de simulation de Monté Carlo a des avantages, cette technique a besoin de l'information complete pour la variable aléatoire en analyse des stabilité du talus, mais it est difficile d'obtenir pratiquement l'information complete à partir de l'investigation sur terrain. C'est pourquoi une méthode d'approximation pour l'analyse de fiabilité a été proposé. La méthode d'approximation telle que first order second moment method (FOSM) et point estimate method (PEM) n'a besoin que du déviance moyen et standard des variables aléatoires. Par conséquent, il est facile d'utiliser quand l'information est limitée. Normalement, une seule forme fermée de la formule pour l'évaluation de la factor de sécurité est necéssitée pour la méthode d'approximation. Puisque la méthode de l'analyse de stabilité pour la rupture en dièdre est compliquée et encombrante, elle ne fournit pas l'équation simple pour l'évaluation de la factor de sécurité. En conséquence, la méthode d'approximation n'est pas pertinente pour la rupture en dièdre. Afin de surmonter cette limitation, une équation simple, obtenue de la méthode d'estimation de probabilité maximum en cas de la ruptur en dièdre, est utilisée de calculer la probabilité de rupture. L'équation simple pour l'estimation directe pour les factors de sécurité de la rupture en dièdre était empiriquement dérivée des talus stables ou en rupture en utilisant l'estimation de probabilité maximum. La technique développée était appliquée à un exemple pratique et les résultats obtenus de la technique développée est comparés aux résultats obtenus de la technique de simulation de Monté Carlo.

Keywords: slope stability, geological hazards, risk assessment, failures, discontinuities

INTRODUCTION

Slope engineering is perhaps the engineering geology subject most dominated by uncertainty since slopes are composed of natural materials (El-Ramly *et al*. 2002). The rock slopes, like any other geotechnical structures, are composed of natural materials, and it means that the inherent uncertainties such as spatial variability are involved. In addition, the uncertainties caused by the limited information are also involved since it is impossible to implement the sufficient number of the geotechnical investigations and laboratory tests. Therefore, the probabilistic approach has been used to deal with the uncertainty effectively. The Monte Carlo simulation method is one of the most widely used probabilistic analyses. This method is very useful especially for problems involving random variables with known or assumed probability distributions. The Monte Carlo simulation involves repeating a simulation process, using in each simulation a particular set of values of the random variables generated in accordance with the corresponding probability distributions. However, the Monte Carlo method requires the great number of the repeated calculations and a great amount of calculation time to evaluate the probability of failure. Moreover, the Monte Carlo simulation technique requires the complete information for random variable such as mean, standard deviation and probability density function (pdf) but it is difficult to acquire the complete information from the limited field investigations and laboratory tests in practice. The information on the random variable is usually limited due to the limitation of sampling numbers.

Therefore, the point estimate method (PEM) has been used to overcome those shortcomings. This is because the point estimate method requires only mean and standard deviation of random variable and obtains the probability of failure with a simple calculation. In addition, the point estimate method is able to consider the correlation between the random variables and adopt the various types of probability density function for random variable. However, in order to utilize the point estimate method in reliability analysis, the simple and closed form of performance function (factor of safety equation in this case) should be provided but the factor of safety equation for rock slope, which is based on the limit equilibrium method, is in a complicated form. Therefore, it is impossible to apply the point estimate method to rock slope stability analysis due to complicated safety factor equation, especially for wedge failure. Therefore, in this study the simple equation for safety factor evaluation is suggested using maximum likelihood estimation. That is, maximum likelihood estimation of the safety factor equation can be obtained as the single formula. Consequently, using the proposed equation, the point estimate method can be applied to rock wedge stability analysis and the probability of slope failure can be evaluated. In the study the practical example is used to evaluate the probability of failure using point estimate method and maximum likelihood estimation. The results are compared to the probability of failure evaluated by Monte Carlo simulation as well as the analysis results obtained from the deterministic analysis.

POINT ESTIMATE METHOD

The point estimate method was proposed by Rosenblueth (1975) and presented by Harr (1981) for the solution of geotechnical problems. The method allows one to use several correlated random variables given by their two or three statistical moments (mean, standard deviation and skewness) to obtain results expressed in terms of the first statistical moments of the examined parameter. Whilst the Monte Carlo method is required the complete probability distributions of the random variables and the considerable computing time, the point estimate method is required only statistical moments of random variables and a simple calculation with moments of random variable. The moments of factor of safety are determined for all possible combinations of point estimate of each random variable. Consequently, the probability of failure is estimated from the moments of factor of safety. The method is initially limited to a three correlated variable analysis and extended to any number of correlated or independent random variables. For instance, to obtain the mean of the factor of safety function for the independent random variables, FS is determined for all possible combinations of one low and one high value (point estimate) of each random variable and the results are weighted by the product of their associated probability concentrations p_{i+} or p_{i-}, and then summed. The entire procedure can be summarized by the following equations:

Given information concerning the first three moments of a random variable, x and the function $FS = FS(x)$, the x_-, x_+, p_-, and p_+ are obtained by

$$p_+ = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{1}{1+[\beta(1)/2]^2}}\right]$$

$$p_- = 1 - p_+$$

$$x_+ = E[x] + \sigma(x)\sqrt{\frac{p_-}{p_+}}$$

$$x_- = E[x] - \sigma(x)\sqrt{\frac{p_+}{p_-}}$$

Then

$$E[FS] = p_- FS(x_-) + p_+ FS(x_+)$$

$$E[FS^2] = p_- FS^2(x_-) + p_+ FS^2(x_+)$$

$$\sigma_{FS} = \sqrt{E[FS^2] - (E[FS])^2}$$

where $\beta(1)$ is coefficient of skewness.

If $\beta(1) = 0$, that is, the probability density function is symmetrical, above equations reduce to

$$p_+ = p_- = \frac{1}{2}$$

$$x_+ = E[x] + \sigma[x]$$

$$x_- = E[x] - \sigma[x]$$

For correlated random variables, additional adjustment must be made to the probability concentrations. Where symmetrically distributed variables are assumed, the point estimates are taken at one standard deviation above and below the expected value, respectively.

For two random variables, four points p_{++}, p_{+-}, p_{-+}, p_{--} are considered and functional relationship is $FS = FS(x_1, x_2)$.

$$p_{++} = p_{--} = \frac{1+\rho}{4}$$

$$p_{+-} = p_{-+} = \frac{1-\rho}{4}$$

$$FS_{\pm\pm} = FS(FS[x_1] \pm \sigma[x_1], FS[x_2] \pm \sigma[x_2])$$

$$E[FS] = p_{++}FS_{++} + p_{+-}FS_{+-} + p_{-+}FS_{-+} + p_{--}FS_{--}$$

$$E[FS^2] = p_{++}FS_{++}^2 + p_{+-}FS_{+-}^2 + p_{-+}FS_{-+}^2 + p_{--}FS_{--}^2$$

where ρ is the coefficient of correlation.

However, a single closed form of performance function is needed for the point estimate method. But the formula for the evaluation of safety factor on the basis of the limit equilibrium method (Hoek & Bray 1981) is elaborate and complicated to use in the point estimate method, especially for wedge failure, in spite of the various readily accessible aids to calculation. Therefore, new formula is required to use the point estimate method in probabilistic slope stability analysis and therefore, the maximum likelihood estimation method of statistics is utilized (Sah *et al*. 1994).

MAXIMUM LIKELIHOOD ESTIMATION METHOD

For maximum likelihood estimation of the slope safety factor, it is necessary to set up a regression equation which is defined as:

$$F_i = \alpha X_i + \beta Y_i$$

in which the constants and are determined as maximum likelihood estimates of the factor of safety F_i and X_i and Y_i are variables defined according to the problem. In order to obtain the simple equation for safety factor calculation, the safety factor is assumed as a random variable and to follow the Gaussian normal distribution with mean, μ_F and standard deviation, σ_F. The probability density function of F_i is given by:

$$f(F_i;\alpha,\beta) = \frac{1}{\sigma_F\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma_F^2}(F_i - \mu_F)^2\right\}$$

The likelihood function L of random samples F_1, F_2, …, F_n of n observations from this population is defined as:

$$L = \prod_{i=1}^{n} [f(F_i;\alpha,\beta)]$$

so,

$$L = \prod_{i=1}^{n} \left[\frac{1}{\sigma_F\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma_F^2}(F_i - \mu_F)^2\right\}\right]$$

Taking logarithms:

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma_F^2 - \frac{1}{2\sigma_F^2}\sum_{i=1}^{n}(\alpha X_i + \beta Y_i - \mu_F)^2$$

Differentiating with respect to α, β and equating to zero we obtain the likelihood equations for estimating α and β are obtained as:

$$\alpha\sum_{i=1}^{n} X_i^2 + \beta\sum_{i=1}^{n} X_i Y_i = \sum_{i=1}^{n} F_i X_i$$

$$\alpha\sum_{i=1}^{n} X_i Y_i + \beta\sum_{i=1}^{n} Y_i^2 = \sum_{i=1}^{n} F_i Y_i$$

Sah *et al*. (1994) proposed the single formula for the safety factor in wedge failure using the above equations and the practical wedge failure cases.

$$F = 0.175\left(\frac{c\cdot \mathrm{cosec}^2\,\psi_p}{\gamma\cdot H\cdot(\cot\psi_p - \cot\psi_f)}\right) + 1.46\,(\cot\psi_p\cdot\tan\phi)$$

where γ is unit weight, c is cohesion, ϕ is internal friction angle, ψ_f is slope angle, ψ_p is the angle of intersection line, and H is slope height. The factors of safety evaluated by the proposed equation are compared with the factors of safety values computed by the limit equilibrium method. The correlation coefficient between two results is 0.957. That is, there is a good agreement between the safety factors calculated by the proposed equation and the safety factors obtained using the limit equilibrium. Therefore, the probabilistic analysis can be implemented with the point estimate method using the above proposed equation. In order to check the applicability of the proposed method in the probabilistic approach, the proposed method was applied to practical examples.

STUDY AREA

The study area consists of an extensive rock cut along Interstate Highway 40 in western North Carolina, USA near the Tennessee border. This area along Interstate 40 shows excellent exposures of a series of metasedimentary rocks of Late Pre-Cambrian age (Figure 1). The site has experienced several large landslides during and after construction. On July 1, 1997, a large rockslide occurred in this area after heavy rainfall, when two discontinuities forming an unstable wedge, failed. An investigation for the relocation of the highway concluded that wedge failures were the most common phenomena. Therefore, a large number of discontinuity orientations and geometries were measured by authors and their random properties were evaluated in this study. Especially, the stability analysis for wedge failure was implemented since the previous research indicated that wedge failure is the major problem in this area.

The slope which has experienced large failure is selected for the stability analysis. The height of the selected slope is approximately 34m and the dip direction and dip of slope are 210° and 75°, respectively.

The discontinuity data were collected from the field and the discontinuity orientations acquired were clustered using the clustering procedure, proposed by Mahtab & Yegulalp (1982). A total 6 discontinuity sets were identified (Figure 2) and their representative orientations and their random properties were listed in Table 1. In addition, discontinuity strength parameters were obtained from direct shear test (Table 1).

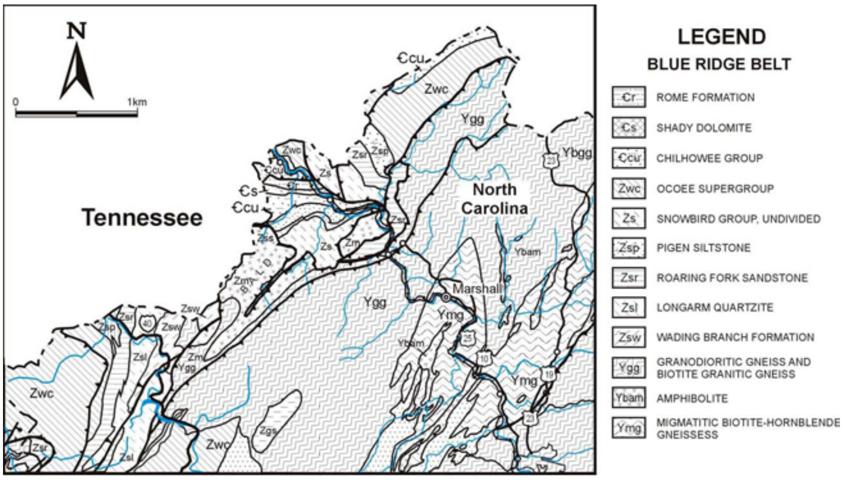

Figure 1. Geological map of the area

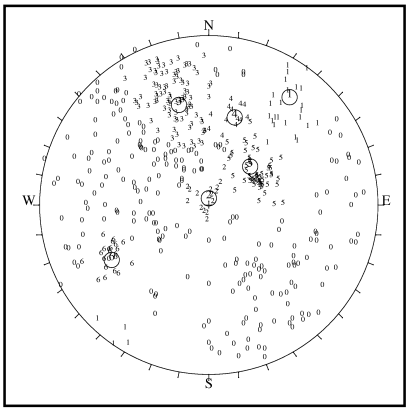

Figure 2. Results of clustering process of discontinuity

EVALUATION FOR PROBABILITY OF SLOPE FAILURE

Results of the deterministic analysis

The deterministic analysis is carried out in order to compare with the stability analysis results obtained from the probabilistic approaches such as the proposed method and Monte Carlo simulation. In the deterministic analysis, mean values of each random variable are selected and the factors of safety are calculated using a fixed single value for each set. In order to analyze the stability of wedge failure, the kinematic analysis for wedge failure should be performed first. For the kinematic analysis, the mean orientations for each discontinuity sets were selected and the kinematic stability was checked for the mean orientation using stereographic projection. Then the factor of safety, suggested by Hoek & Bray (1981) was evaluated to examine the kinetic stability. In the results of the deterministic analysis, the combination of J1 & J3, J1 & J4, J1 & J6, J3 & J4, J3 & J5, J4 & J5 are evaluated as kinematically unstable and thus the factors of safety are evaluated for the discontinuity combinations which are determined as kinematically unstable (Table 2). For the combinations which are analyzed as kinematically stable, the kinetic analysis is not performed. The analysis is carried out on the basis of the assumption for the dry condition.

Results of Monte Carlo simulation

The Monte Carlo simulation is one of the most widely used probabilistic analysis and therefore, the analysis technique and procedure are well developed and known (Kim & Major 1978, Muralha 1991, Muralha & Trunk 1993, Nilsen 2000, Park & West 2001, Pathak & Nilsen 2004, Park *et al*. 2005). The Monte Carlo simulation generated random numbers repeatedly for random variables, and evaluated the performance function with randomly generated values. In this study, the possibility of kinematic instability is evaluated first. The probabilistic kinematic analysis acquires the probability of kinematic instability using random properties of discontinuity orientations, and their results are different from the results obtained from the deterministic analysis. This is because the scatter of discontinuity orientation is considered in the probabilistic analysis and the scattered orientations show the possibility of the kinematic instability even though the mean orientation of each set does not indicate kinematic instability. In contrast, only mean values for each discontinuity are considered and evaluated in the deterministic analysis. Consequently the combinations of J1 & J2, J1 & J5, J2 & J5 and J5 & J6 were analyzed as kinematically stable in the deterministic analysis but as unstable in the probabilistic analysis. For the joint combinations evaluated as kinematically unstable, the kinetic analysis is carried out. For the probabilistic kinetic analysis, Hoek & Bray (1981)'s safety factor equation is used as the performance function, and the great number of factors of safety is generated in the Monte Carlo simulation process. The number of factor of safety less than 1 is counted and the probability of failure is evaluated. In the probabilistic kinematic analysis, only the combinations of J2 & J3, J2 & J4, J2 & J6 and J3 & J6 are evaluated as kinematically stable and therefore, their probabilities of failure were not considered. As can be observed in Table 2, the probabilities of kinetic instability range from 1.3% to 56.1%. Especially, the combinations involved with J5, such as J1 & J5, J2 & J5, J3 & J5, J5 & J6 show relatively high probabilities.

Results of Point Estimate Method and Maximum Likelihood Estimation Method

As mentioned in the previous paragraph, the point estimate method is suggested to overcome the shortcomings of Monte Carlo simulation method. However, in the point estimate method, the simple formula is a requisite for the analysis. Especially the safety factor equation for wedge failure is quite complicated and thus it is impossible to apply the point estimate method to wedge failure problem.

Table 1. Random properties of discontinuity sets

Set I.D.	J1	J2	J3	J4	J5	J6	PDF
Mean orientation (dip direction/dip)	217/77	183/05	163/63	196/56	227/37	061/66	Fisher
Fisher constant	42	53	29	119	36	106	
Mean friction angle (degree)	40	30	30	30	27.03	30	Normal
STD of friction angle	3.78	3.0	3.0	3.0	2.94	3.0	

Table 2. Analysis results

Set No. 1	Set No. 2	Factor of safety	Probability of failure		
			kinematic instability	kinetic instability	
				using MC	using PEM and MLE
J1	J2	Stable	0.001	0.036	0.021
J1	J3	0.29	0.4	0.013	0.001
J1	J4	0.32	0.374	0.007	0
J1	J5	Stable	0.037	0.244	0.384
J1	J6	0.43	0.189	0.007	0.001
J2	J3	Stable	0	0	0
J2	J4	Stable	0	0	0
J2	J5	Stable	0.002	0.331	0.211
J2	J6	Stable	0	0	0
J3	J4	0.09	0.935	0.013	0.016
J3	J5	0.36	0.738	0.561	0.502
J3	J6	Stable	0	0	0
J4	J5	0.33	0.471	0.514	0.563
J4	J6	0.39	0.309	0	0
J5	J6	Stable	0.009	0.193	0.214

Therefore, the maximum likelihood estimation is utilized to propose the single safety factor formula in wedge failure.

Consequently, the point estimate method can be used to evaluate the probability of failure with the proposed single formula. In this procedure, the probabilities of kinetic instability for each discontinuity combinations were evaluated. Then the probabilities of kinetic instability obtained from the proposed approach were compared to those from Monte Carlo simulation method. The analysis results are listed in Table 2. The analysis results obtained from the proposed approach are somewhat different from the results evaluated from the Monte Carlo simulation method. This is because the point estimate method and maximum likelihood estimate method are the approximation approach. That is, in order to overcome the shortcomings of the previous method, the approximation approach is used and it affects the analysis results. In spite of the shortcomings of the approximate method, the approach can be readily applied in practice without the complicated calculations and the reliable results can be acquired.

CONCLUSIONS

Recently as the engineering geologists have an interest in uncertainty in slope engineering, the interests in the probabilistic approach are increased. Especially for rock slope, the random properties of the discontinuity as well as the random properties of intact rock is one of the most important parameters in slope stability analysis. Therefore, the probabilistic approach is widely used in rock slope stability analysis, substituting for the traditional deterministic analysis. The Monte Carlo simulation method is one of the most widely used probabilistic analyses but it requires the complete information for random variable and a lot of computing time and efforts. Therefore, the point estimation method as one of the approximation approaches, is suggested since the point estimation method requires only mean and standard deviation of random variable and a simple calculation. However, the point estimation method requires a singular formation of factor of safety evaluation and the factor of safety equation for wedge failure is quite complicated. Consequently point estimation method cannot be applied to wedge failure.

In this study maximum likelihood estimation is used to obtain simple equation for wedge failure. Therefore, using a new simple equation, the point estimation method is applied to wedge failure. In order to check the applicability of the proposed method, the method was applied to practical examples. In addition, using the Monte Carlo simulation, the probabilistic analysis is carried out to compare with the analysis results of the proposed method. The analysis results obtained from the proposed approach are somewhat different from the results evaluated from the Monte Carlo method. This is because the point estimate method and maximum likelihood estimate method are the approximation approach. That is, in order to overcome the shortcomings of the previous method, the approximation approach is used and it affects the analysis results. In spite of the shortcomings of the approximate method, the approach can be readily applied in practice without the complicated calculations and consequently the reliable results can be acquired.

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REFERENCES

EL-RAMLY, H., MORGENSTERN, N.R. & CRUDEN, D.M. 2002. Probabilistic slope stability analysis for practice. *Canadian Geotechnical Journal,* **39**, 665-683.

HARR, M.E. 1987. *Reliability Based on Design in Civil Engineering*. McGraw-Hill, New York.

HOEK, E.T. & BRAY, J.W. 1981. *Rock Slope Engineering*. Institute of Mining and Metallurgy. London.

KIM, H. & MAJOR, G. 1978. Application of Monte Carlo techniques to slope stability analyses. *Proceedings of 19th U.S. Symposium on Rock Mechanics*, Nevada, 28-38.

MAHTAB, M.A. & YEGULALP, T.M. 182. A rejection criterion for definition of clusters in orientation data. *Proceedings Of 22nd Symposium on Rock Mechanics*, New York, 116-123.

MURALHA, J. 1991. A probabilistic approach to the stability of rock slope. *7th Congress of ISRM*, Germany, A.A. Balkema, 921-927.

MURALHA, J. & TRUNK, U. 1993. Stability of rock blocks evaluation of failure probabilities by Monte Carlo simulation and first order reliability methods. *International Symposium on Assessment and Prevention of Failure Phenomena in Rock Engineering*, A.A. Balkema, 759-765.

NILSEN, B. 2000. New trend in rock slope stability analysis. *Bulletin of Engineering Geology and the Environments*, **58**, 173-178.

PARK, H.J. & WEST, T.R. 2001. Development of a probabilistic approach for rock wedge failure. *Engineering Geology*, **59**, 233-251.

PARK, H.J., WOO, I. & WEST, T.R. 2005. Probabilistic analysis of rock slope stability and random property parameters, Interstate Highway 40, Western North Carolina. *Engineering Geology*, **79**, 230-250.

PATHAK, S. & NILSEN, B. 2004. Probabilistic rock slope stability analysis for Himalayan condition. *Bulletin of Engineering Geology and the Environments*, **63**, 25-30.

ROSENBLUETH, E. 1975. Point estimates for probability moments. *Proceedings of National Academy of Sciences*, **72**, No. 10.

SAH, N.K., SHEOREY, P.R. & UPADHYAYA, L.N. 1994. Maximum likelihood estimation of slope stability. *International Journal of Rock Mechanics and Mining & Geomechanical Abstract,* **31**(1), 47-53